FROM COHERENT TO FINITENESS SPACES

PIERRE HYVERNAT

Laboratoire de Mathématiques, Université de Savoie, 73376 Le Bourget-du-Lac Cedex, France.
e-mail address: Pierre.Hyvernat@univ-savoie.fr

Abstract. This short note presents a new relation between coherent spaces and finiteness spaces. This takes the form of a functor from **Coh** to **Fin** commuting with the additive and multiplicative structure of linear logic. What makes this correspondence possible and conceptually interesting is the use of the infinite Ramsey theorem. Along the way, the question of the cardinality of the collection of finiteness spaces on $\mathbf{N}$ is answered.

Basic knowledge about coherent spaces and finiteness spaces is assumed.

Introduction

The category of coherent spaces was the first denotational model for linear logic [4]: the basic objects are countable reflexive non-oriented graphs, and we are more specifically interested by their *cliques* (complete subgraphs). If C is such a graph, we write $\mathcal{C}(C)$ for the collection of its cliques. Coherent spaces enjoy a rich algebraic structure where the important operations are:

- the (reflexive closure of the) complement, written $C_1^\perp$;
- the product, written $C_1 \otimes C_2$;
- the disjoint union, written $C_1 \oplus C_2$.

If one forgets about edges and only looks at vertices, the corresponding operations are simply the identity, the usual cartesian product "$\times$" and the disjoint union "$\oplus$".

More recently, T. Ehrhard introduced the notion of finiteness space [2] to give a model for the differential λ-calculus [3], which can be seen as an enrichment of linear logic. The point that interests us most here is that the collection of *finitary sets* of a finiteness space is closed under finite unions. This is definitely not the case with the cliques of a coherent space. This property is crucial for the interpretation of non-deterministic sums of terms [2, 6], which correspond in the models to linear combinations of simple terms. In this note, we only look at a qualitative version, where coefficients play no role. In other words, coefficients live in the *rig* (ring without *n*egatives) $\{0, 1\}$ with $1 + 1 = 1$.

1998 ACM Subject Classification: F.4.1, F.3.2.
Key words and phrases: Linear Logic, Coherent Space, Finiteness Space, Ramsey Theorem.
This work has been partially funded by the French project choco (`ANR-07-BLAN-0324`).

LOGICAL METHODS
IN COMPUTER SCIENCE

DOI:10.2168/LMCS-???

Very briefly, a finiteness space is given by a countable set $|\mathcal{F}|$, called its *web*, and a collection $\mathcal{F}$ of subsets of $|\mathcal{F}|$, called the *finitary sets*, which satisfies

$$\mathcal{F}^{\perp\!\!\perp\perp\!\!\perp} \quad = \quad \mathcal{F}$$

with

$$\mathcal{D}^{\perp\!\!\perp} \quad \stackrel{\text{def}}{=} \quad \big\{x \;\big|\; \forall y \in \mathcal{D}, \#(x \cap y) < \aleph_0\big\}$$

where $\#(x)$ is the cardinal of x, and $\aleph_0$ is the least infinite cardinal. In natural language, the crucial property is that whenever $x \in \mathcal{F}$ and $y \in \mathcal{F}^{\perp\!\!\perp}$, the intersection $x \cap y$ is finite. Algebraic constructions similar to the ones for coherent spaces can be defined on finiteness spaces, and they are characterized by:

- the dual $\mathcal{F}^{\perp\!\!\perp}$;
- the tensor $\mathcal{F}_1 \otimes \mathcal{F}_2 \stackrel{\text{def}}{=} \{r \mid \pi_1(r) \in \mathcal{F}_1,\ \pi_2(r) \in \mathcal{F}_2\}$;
- the coproduct $\mathcal{F}_1 \oplus \mathcal{F}_2 \stackrel{\text{def}}{=} \{x_1 \uplus x_2 \mid x_1 \in \mathcal{F}_1,\ x_2 \in \mathcal{F}_2\}$.

Here again, if one forgets about finitary sets and only looks at the webs of finiteness spaces, the corresponding operations are just the identity, the usual cartesian product and the disjoint union.

Remarks.

(1) The operations on finiteness spaces are actually defined in a way that makes it clear that they yield finiteness spaces. They are then proved to be equivalent to the definitions given above [2].
(2) Any operator of the form $\mathcal{X} \mapsto \mathcal{X}^* = \{y \mid \forall x \in \mathcal{X},\ R(x,y)\}$ is contravariant with respect to inclusion and yields a *closure operator* when applied twice. In particular, any set of the form $\mathcal{Y} = \mathcal{X}^*$ satisfies $\mathcal{Y} = \mathcal{Y}^{**}$.
(3) There is another presentation of coherent spaces that closely matches the definition of finiteness spaces: a coherent space is given by a collection $\mathcal{C}$ of subsets of $|\mathcal{C}|$ which satisfy $\mathcal{C}^{**} = C$, where $\mathcal{D}^* = \{x \mid \forall y \in \mathcal{D}, \#(x \cap y) \leq 1\}$.

1. From Coherence to Finiteness

The idea is rather simple: we would like to close the collection of cliques of a coherent space under finite unions. Unfortunately (but unsurprisingly), the notion of "finite unions of cliques" is not very well behaved, especially with respect to the dual. Recall that an anticlique of C (also called independent, or stable sets) is a clique in $C^\perp$. We consider the following notion:

Definition 1.1. If C is a coherent space, we call a subset of $|C|$ *finitely incoherent* if it doesn't contain infinite anticliques. We write $\mathcal{F}(C)$ for the collection of all finitely incoherent subsets of C.

The next lemma follows directly from the definition.

Lemma 1.2.

(1) *Any finite subset of $|C|$ is finitely incoherent;*
(2) *a subset of a finitely incoherent subset is finitely incoherent;*
(3) *finitely incoherent subsets are closed under finite unions;*
(4) *any clique is finitely incoherent.*

Note however that a finitely incoherent set needs not be a finite union of cliques: take for example the graph composed of the disjoint union of all the complete graphs K_n for $n \geq 1$. This graph doesn't contain an infinite clique, but it is not a finite union of anticliques; so, its dual is finitely incoherent but is not a finite union of cliques.

The next lemma is more interesting as it implies that the collection of finitely incoherent subsets forms a finiteness space.

Lemma 1.3. *If C is a coherent space, we have:*

$$\mathcal{C}(C)^{\perp\!\!\!\perp} \quad = \quad \mathcal{F}(C^\perp) \ .$$

Proof.

($\subseteq$) Let x be in $\mathcal{C}(C)^{\perp\!\!\!\perp}$, and suppose, by contradiction, that x is not in $\mathcal{F}(C^\perp)$, i.e., x contains an infinite anticlique y of $C^\perp$. This set y is a clique in C, i.e., $y \in \mathcal{C}(C)$. Since $x \cap y = y$ is infinite, this contradicts the hypothesis that $x \in \mathcal{C}(C)^{\perp\!\!\!\perp}$.

($\supseteq$) Let x be finitely incoherent in $C^\perp$, i.e., x doesn't contain an infinite clique of C; let y be in $\mathcal{C}(C)$. Since $x \cap y \in \mathcal{C}(C)$ and $x \cap y$ is contained in x, it cannot be infinite. This shows that $x \in \mathcal{C}(C)^{\perp\!\!\!\perp}$.

□

By remark 2 on page 2, we thus get the expected corollary:

Corollary 1.4. *If C is a coherent space, then $\mathcal{F}(C)$ is a finiteness space.*

What was unexpected is the following:

Lemma 1.5. *If C is a coherent space, then:*

$$\mathcal{F}(C^\perp) \quad = \quad \mathcal{F}(C)^{\perp\!\!\!\perp} \ .$$

Proof. Because of the previous lemma, and because $\_^{\perp\!\!\!\perp}$ is contravariant with respect to inclusion, we only need to show that $\mathcal{C}(C)^{\perp\!\!\!\perp} \subseteq \mathcal{F}(C)^{\perp\!\!\!\perp}$. Suppose that $x \in \mathcal{C}(C)^{\perp\!\!\!\perp}$, and let $y \in \mathcal{F}(C)$; we need to show that $x \cap y$ is finite.

- Since $x \cap y \subseteq y \in \mathcal{F}(C)$, $x \cap y$ cannot contain an infinite anticlique;
- since $x \cap y \subseteq x \in \mathcal{C}(C)^{\perp\!\!\!\perp}$, $x \cap y$ cannot contain an infinite clique.

Those two points imply, by the infinite Ramsey theorem,[1] that $x \cap y$ is finite.

□

The other linear connectives are similarly behaved with respect to the notion of finitely incoherent sets. We have:

Lemma 1.6. *If C_1 and C_2 are coherent spaces, then we have both*

$$\mathcal{F}(C_1 \oplus C_2) \quad = \quad \mathcal{F}(C_1) \oplus \mathcal{F}(C_2) \ ,$$

and

$$\mathcal{F}(C_1 \otimes C_2) \quad = \quad \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$$

where the connectives on the left are the coherent spaces' ones, and the connectives on the right are the finiteness spaces' ones.

[1] *Infinite Ramsey theorem:* suppose G is a countably infinite set, then, for every assignment of c colors to the subsets of G of cardinality n, there is an infinite $I \subseteq G$ s.t. all subsets of I of cardinality n have the same color. (Refer to [5] or one of the many textbooks on combinatorics covering it.) For $n = 2$ and $c = 2$, it amounts to "each countably infinite graph has an infinite clique or an infinite anticlique".

Proof. The $\oplus$ part is direct; for the $\otimes$ part, recall that $r \in \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$ is equivalent to $\pi_1(r) \in \mathcal{F}(C_1)$ and $\pi_2(r) \in \mathcal{F}(C_2)$.

($\subseteq$) Suppose r doesn't contain an infinite anticlique; neither $\pi_1(r)$ nor $\pi_2(r)$ can contain an infinite anticlique, as it would imply the existence of an infinite anticlique in r.

($\supseteq$) Suppose that $r \in \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$ contains an infinite anticlique r' of $C_1 \otimes C_2$. At least one of $\pi_1(r')$ or $\pi_2(r')$ must be infinite, otherwise, r' itself would be finite. Suppose $\pi_1(r')$ is infinite; because $\pi_1(r') \subseteq \pi_1(r)$, it cannot contain an infinite anticlique. By the infinite Ramsey theorem, it thus contains an infinite clique x. For each $a \in x$, chose one element b inside the fiber $r'(a) = \{b \mid (a,b) \in r'\}$. Two such b's cannot be coherent as it would contradict the fact that r' is an anticlique. In particular, all such b's are distinct. We have constructed an infinite anticlique in $\pi_2(r') \subseteq \pi_2(r) \in \mathcal{F}(C_2)$. Contradiction!

□

For finiteness spaces, the operation $\oplus$ coincides with its dual [2]. In particular, for finiteness spaces coming from from coherent spaces, we have

$$\big(\mathcal{F}(C_1) \oplus \mathcal{F}(C_2)\big)^{\perp\!\!\!\perp} \quad = \quad \mathcal{F}(C_1)^{\perp\!\!\!\perp} \oplus \mathcal{F}(C_2)^{\perp\!\!\!\perp} \ .$$

Lemmas 1.5 and 1.6 thus imply that

$$\mathcal{F}\left((C_1 \oplus C_2)^\perp\right) \quad = \quad \mathcal{F}\left(C_1^\perp \oplus C_2^\perp\right) \ .$$

The direct proof of this equality is also quite easy.

Both coherent spaces and finiteness spaces form categories, where:

- a morphism from C to D in **Coh** is a clique in $(C \otimes D^\perp)^\perp$,
- a morphism from $\mathcal{F}$ to $\mathcal{G}$ in **Fin** is a finitary set in $(\mathcal{F} \otimes \mathcal{G}^{\perp\!\!\!\perp})^{\perp\!\!\!\perp}$.

In both cases, morphisms are special relations between webs and composition is the usual composition of relations:

$$r \circ s \quad \stackrel{\mathsf{def}}{=} \quad \big\{(a,c) \ \big|\ \exists b\, (a,b) \in s \text{ and } (b,c) \in r\big\} \ .$$

From all the above, we can conclude that:

Proposition 1.7. *The operation $\mathcal{F}(\_)$ can be lifted to a functor from* **Coh** *to* **Fin***:*

(1) *it sends C to $\mathcal{F}(C)$*
(2) *and $r \in \mathbf{Coh}[C,D]$ to $r \in \mathbf{Fin}[\mathcal{F}(C), \mathcal{F}(D)]$.*

Moreover, this functor commutes with $\_^\perp$, $\_ \otimes \_$ and $\_ \oplus \_$.

This functor is faithful (but not full), and it is not injective on objects as adding or removing any finite number of edges to a coherent space doesn't change its image via $\mathcal{F}(\_)$. Moreover, this functor commutes with the forgetful functors from **Coh** and **Fin** to **Rel**, the category of sets and relations.

In a sense, coherent spaces allow one to define a collection of simple finiteness spaces. An informal argument regarding this simplicity can be found in the following remark: the logical complexity of the formula expressing "$x \in \mathcal{A}^{\perp\!\!\!\perp\perp\!\!\!\perp}$", i.e., "$x$ is finitary with respect to $\mathcal{A}$" changes when $\mathcal{A}$ comes from a coherent space. If we write $y \subsetneq_\infty x$ for "y is an infinite subset of x", we have [2]:

$$x \in \mathcal{A}^{\perp\!\!\!\perp} \qquad \Longleftrightarrow \qquad \forall y \subsetneq_\infty x \quad y \notin \mathcal{A}$$

whenever $\mathcal{A}$ is downward closed. Thus:

$$x \in \mathcal{A}^{\perp\!\!\perp\perp\!\!\perp} \qquad \Longleftrightarrow \qquad \forall y \subsetneq_\infty x \quad \exists z \subsetneq_\infty y \quad z \in \mathcal{A}\ .$$

Because $y \subsetneq_\infty x$ is a Σ^1_1-formula (no universal second-order quantifiers), "$x \in \mathcal{A}^{\perp\!\!\perp\perp\!\!\perp}$" is a Π^1_2-formula (second-order quantifiers are $\forall\exists$). Note that even if $\mathcal{A}$ isn't downward closed, the formula expressing "$x \in \mathcal{A}^{\perp\!\!\perp\perp\!\!\perp}$" is still a Π^1_2-formula. For the particular case when $\mathcal{A}$ is the set of cliques of a coherent spaces C, we obtain

$$x \in \mathcal{A}^{\perp\!\!\perp\perp\!\!\perp} \qquad \Longleftrightarrow \qquad x \in \mathcal{C}(C^\perp)^{\perp\!\!\perp} \qquad \Longleftrightarrow \qquad \forall y \subsetneq_\infty x \quad \exists\, a, b \in y \quad (a,b) \in C$$

which is only a Π^1_1-formula.

2. Cardinality of Finiteness Spaces

So, coherent spaces can be used to define a collection of finiteness spaces closed under the linear operations ($\_^\perp$, $\_ \otimes \_$ and $\_ \oplus \_$). It is natural to ask whether all finiteness spaces can be obtained in this way. The previous informal remark about the logical complexity of coherence versus finiteness points toward a negative answer. Here is a formal proof which also answers a question raised by T. Ehrhard:

Proposition 2.1. *If A is infinite countable, the cardinality of finiteness spaces on A is exactly that of $\mathcal{P}(\mathcal{P}(A))$. The cardinality doesn't change if we consider finiteness spaces up-to isomorphisms, i.e., up-to permutations of A.*

Since the cardinality of coherent spaces on A is the same as that of $\mathcal{P}(A \times A) \simeq \mathcal{P}(A)$, we can conclude that:

Corollary 2.2. *If A is infinite countable, there are strictly more finiteness spaces on A than coherent spaces on A.*

Proof. Let A be infinite countable; up-to isomorphism, we can assume that $A = \mathbf{B}^{<\omega}$, the set of finite sequences of bits. If x is an *infinite* sequence of bits, write $x^\downarrow$ for the set of finite approximations of x; and if X is a set of such "real numbers", write $X^\downarrow$ for the set $\{x^\downarrow \mid x \in X\}$. We have $X^\downarrow \subset \mathcal{P}(A)$ for any such set X.

Suppose now that $X \neq X'$ with, for example, $x \in X$ but $x \notin X'$. Since $x^\downarrow$ is infinite and $x^\downarrow \in X^\downarrow$, we have $x^\downarrow \notin X^{\downarrow\perp\!\!\perp}$. However since two different reals must differ on some finite approximation, we have that $x^\downarrow \in X'^{\downarrow\perp\!\!\perp}$. Thus, the finiteness spaces $(A, X^{\downarrow\perp\!\!\perp})$ and $(A, X'^{\downarrow\perp\!\!\perp})$ differ.

This defines an injective map $X \mapsto (A, X^{\downarrow\perp\!\!\perp})$ from arbitrary sets of reals to finiteness spaces on A. This shows that finiteness spaces on A have at least the same cardinality as $\mathcal{P}(\mathbf{R}) \simeq \mathcal{P}(\mathcal{P}(A))$. Since it cannot be more than that, we have equality.

An isomorphism in the category **Fin** is a particular relation with a left and right inverse. This implies that it is in fact the graph of a bijection, and thus, two finiteness spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ on a set A are isomorphic if and only if there is a bijection $\sigma : A \to A$ such that

$$\mathcal{F}_2 \quad = \quad \sigma \cdot \mathcal{F}_1 \stackrel{\text{def}}{=} \big\{\sigma(x) \ \big|\ x \in \mathcal{F}_1\big\}$$

where $\sigma(x)$ when $x \subseteq A$ is simply the direct image of the set x. Because the cardinality of each equivalence class is at most $\#\big(\mathcal{P}(A)\big)$ (this is the cardinality of permutations on A), and since $\kappa \times \#\big(\mathcal{P}(A)\big) = \max\big(\kappa, \#(\mathcal{P}(a))\big)$, there must be at least $\#\big(\mathcal{P}(\mathcal{P}(A))\big)$ such equivalence classes to cover the whole collection of finiteness spaces.

The cardinality of finiteness spaces on A up-to isomorphism is thus the same as that of finiteness spaces on A up-to plain equality: $\#\big(\mathcal{P}(\mathcal{P}(A))\big)$.

□

It is slightly interesting to note that the same reasoning doesn't apply to higher cardinalities since $\#(A^{<\omega}) = \#(A)$ if A is uncountable.

Conclusion

The situation with respect to full linear logic isn't totally clear. We have a base category **FinCoh** with:

- coherent spaces as objects
- and finitely incoherent linear maps as morphisms: $\mathbf{FinCoh}[C, D] = \mathcal{F}\left((C \otimes D^\perp)^\perp\right)$.

This category is a linear, full subcategory of the category **Fin** of finiteness spaces.

Lifting the usual set-based notion of exponentials for coherent spaces to this category is impossible: because the web of $!C$ is the collection of finite cliques of C (uniformity), this construction isn't even functorial. Take for example K_n and $K_n^\perp$: since their sets of vertices have the same finite cardinality, they are isomorphic in **FinCoh**. However $!K_n$ and $!(K_n^\perp)$ have sets of vertices of different cardinalities, namely 2^n and $n+1$: they cannot be isomorphic in **FinCoh**.[2]

The multiset-based notion of exponentials, where one defines the web $|!C|$ to be the collection of finite multisets whose support is a clique, doesn't seem to help. It is not functorial in any canonical way, as shown by the same example of $K_n \simeq K_n^\perp$: the corresponding sets of vertices for $!K_n$ and $K_n^\perp$ are $\mathcal{M}_f\{1, \dots, n\}$ and $\mathcal{M}_f\{1\} \cup \dots \cup \mathcal{M}_f\{n\}$. Note that in both cases, the collection of finitely incoherent sets consists of *all* the subsets of the web, so that a non-canonical isomorphism is still possible. However, if such an isomorphism exists, it doesn't commute to the forgetful functors to the category of sets and relations.

Using the non-uniform coherent spaces [1] isn't a solution either as it introduces a third relation: neutrality. A non-uniform coherent space is thus a non-oriented graph with two kinds of edges: strict coherence edges and neutral edges. Neutral edges are left unchanged by duality and we take the complement of the rest. In the usual coherent spaces, the only neutral edges are the loops around vertices. The natural notion is to define a clique as a set of vertices that are pairwise coherent or neutral, but one could also consider "strict" cliques. Thus, there are two possible definitions of $\mathcal{C}(C)$:

$$\mathcal{C}(C) \quad \stackrel{\text{def}}{=} \quad \begin{cases} \text{cliques (pairwise coherence or neutrality)} \\ \text{strict cliques (pairwise strict coherence)} \end{cases} \quad ,$$

and similarly, there are two possible definitions of $\mathcal{F}(C)$:

$$\mathcal{F}(C) \quad \stackrel{\text{def}}{=} \quad \begin{cases} x\text{'s that do not contain anticliques} \\ x\text{'s that do not contain strict anticliques} \end{cases} \quad .$$

None of the four possibilities enjoys the adequate properties. When using cliques and anticliques, point 4 of Lemma 1.2 fails: we do not have $\mathcal{C}(G) \subseteq \mathcal{F}(C)$. The cliques/strict anticliques and strict cliques/anticliques versions fail at Lemma 1.3: one inclusion or the

[2] An isomorphism in **FinCoh** is in particular an isomorphism in **Rel** which is an isomorphism in **Set**.

other doesn't hold. With the strict versions of $\mathcal{C}(C)$ and $\mathcal{F}(C)$, we go as far as the proof of Lemma 1.5. However,

- $x \cap y$ doesn't contain an infinite strict anticlique
- and $x \cap y$ doesn't contain an infinite strict clique

only implies, by the infinite Ramsey theorem for three colors, that $x \cap y$ is finite *or contains an infinite set of pairwise neutral vertices.*

Finding an appropriate notion of exponential to extend this category to a model of the algebraic λ-calculus, or better yet, of the differential λ-calculus is thus left open at this point.

References

1. Antonio Bucciarelli and Thomas Ehrhard, *On phase semantics and denotational semantics: the exponentials*, Annals of Pure and Applied Logic **109** (2001), no. 3, 205–241.
2. Thomas Ehrhard, *Finiteness spaces*, Mathematical Structures in Computer Science (2005), 615–646.
3. Thomas Ehrhard and Laurent Regnier, *The differential lambda-calculus*, Theoretical Computer Science **309** (2003), no. 1–3, 1–41.
4. Jean-Yves Girard, *Linear logic*, Theoretical Computer Science **50** (1987), 1–102.
5. Frank Plumpton Ramsey, *On a problem of formal logic*, Proceedings of The London Mathematical Society **s2-30** (1930), 264–286.
6. Lionel Vaux, *The algebraic lambda calculus*, Mathematical Structures in Computer Science **19** (2009), no. 5, 1029–1059.